

# TOEPLITZ OPERATORS AND THE ROE-HIGSON TYPE INDEX THEOREM IN RIEMANNIAN SURFACES

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**ABSTRACT.** We study a two dimensional analogue of the Roe-Higson index theorem for a partitioned manifold. We prove that Connes' pairing of some invertible element with Roe's cyclic one-cocycle coincides to the Fredholm index of a Toeplitz operator. In the proof of this paper, we use some properties of a circle and use Higson's argument. In the last section, there is a example of partitioned manifold, which is not a cylinder, with non-trivial pairing.

## INTRODUCTION

Let  $M$  be a partitioned complete Riemannian manifold, that is, there exist codimension zero submanifolds with boundary  $M^\pm$  and a codimension one closed submanifold  $N$  such that  $M = M^+ \cup M^-$  and  $N = M^+ \cap M^- = \partial M^+ = \partial M^-$ . Let  $S \rightarrow M$  be a Clifford bundle in the sense of [12, Definition 3.4] and let  $D$  be the Dirac operator of  $S$ . Let  $S_N$  be the restriction on  $N$  of  $S$ . Then we can assume  $S_N$  is a graded Clifford bundle over  $N$ . Let  $D_N$  be the graded Dirac operator of  $S_N$ . In the above setting, we denote by  $u_D$  the Cayley transform of  $D$ , that is,  $u_D := (D - i)(D + i)^{-1}$ . Then  $u_D$  is a invertible element in the Roe algebra  $C^*(M)$ . In [11], Roe defined the odd index class  $\text{odd-ind}(D) \in K_1(C^*(M))$ , which is represented by  $u_D$ , and he also defined the cyclic one-cocycle  $\zeta$ , which is called the Roe cocycle, on a dense subalgebra of  $C^*(M)$ . Then Connes' pairing of cyclic cohomology with  $K$ -theory  $\langle u_D, \zeta \rangle$  is agree up to constant with the Fredholm index of  $D_N^+$  [11]. In [8], Higson gave a very simple and clear proof of this theorem. So we call that this theorem is the Roe-Higson index theorem.

When  $M$  is even dimensional,  $\text{index}(D_N^+)$  is always zero in the Roe-Higson index theorem. So we want to get another non trivial formula. On the other hand, Toeplitz operators play a role of fundamental operators in the Atiyah-Singer index theorem on odd dimensional closed manifolds [1, §20, 24]. Therefore we shall prove another index theorem with Toeplitz operators on  $N$ . Namely, we shall prove Connes' pairing of the Roe cocycle with another  $K_1$ -element  $[u_\phi]$  (see Proposition 1.4) is agree with the Fredholm index of a Toeplitz operator on  $N$  (Theorem 1.9).

**Main Theorem .** *Let  $M$  be a partitioned oriented Riemannian surface. Let  $S$  be a graded spin bundle over  $M$  with  $\mathbb{Z}_2$ -grading  $\epsilon$  and let  $D$  be the graded Dirac operator on  $M$ . We assume  $\phi \in C^1(M; GL_l(\mathbb{C}))$  satisfies  $\|\phi\| < \infty$ ,  $\|\text{grad}(\phi)\| < \infty$  and  $\|\phi^{-1}\| < \infty$ . Define*

$$u_\phi := (D + \epsilon)^{-1} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} (D + \epsilon).$$

*Then the following formula holds:*

$$\langle [u_\phi], \zeta \rangle = -\frac{1}{8\pi i} \text{index}(T_\phi).$$

The main strategy of the proof of our main theorem is to reduce the general two dimensional case to the  $\mathbb{R} \times S^1$  case by similar argument of Higson in [8]. To prove the  $\mathbb{R} \times S^1$  case, we can use the family of very useful functions  $\{e^{ikx}\}_k$ , which is orthonormal basis of  $L^2(S^1)$ , all eigenvectors of Dirac operator  $-i\partial/\partial x$  on  $S^1$ , and all representative elements of fundamental group  $\pi_1(S^1)$ .

The general dimensional case of our main theorem is in [13]. It contains the  $KK$ -theoretic construction of  $[u_\phi]$ .

## 1. MAIN THEOREM

**1.1. Elements of the  $K_1$  group.** In this subsection, we define  $K_1$  elements used in our main theorem.

**Definition 1.1.** *Let  $M$  be a oriented complete Riemannian manifold. We assume that the triple  $(M^+, M^-, N)$  satisfies the following conditions:*

- $M^+$  and  $M^-$  are two codimension zero submanifolds of  $M$  with boundary,
- $M = M^+ \cup M^-$ ,
- $N$  is a codimension one closed submanifold of  $M$ ,
- $N = M^+ \cap M^- = -\partial M^+ = \partial M^-$ .

*Then we call that  $(M^+, M^-, N)$  is a partition of  $M$ . Then we call that  $M$  is a partitioned manifold.*

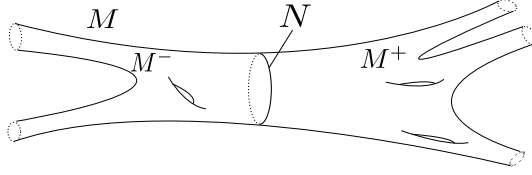


FIGURE 1. Partitioned manifold

In this paper, we assume that  $M$  is a partitioned oriented two dimensional complete Riemannian manifold (i.e. complete Riemannian surface) and  $(M^+, M^-, N)$  is a partition of  $M$ . Let  $S$  be a graded spin bundle of  $M$ <sup>1</sup> with grading  $\epsilon$  and a Clifford action  $c$ . Let  $D$  be the graded Dirac operator of  $S$ . For simplicity, we assume that  $M$  is connected and  $N$  is isometric to the unit circle  $S^1$ . We also assume  $c(d/dt)c(d/dx) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$  on  $(-\epsilon, \epsilon) \times N$  of tubular neighborhood of  $N$ , where  $\{d/dt, d/dx\}$  is a positively oriented orthonormal vector fields on  $(-\epsilon, \epsilon) \times N$ .

**Remark 1.2.** *In particular, we assume  $M$  is non compact, then  $S$  is trivial bundle:  $S = M \times \mathbb{C}^2$ . Especially, if  $M = \mathbb{R}^2 \cong \mathbb{C}$  with standard metric, then*

$$D = 2 \begin{bmatrix} 0 & -\partial/\partial \bar{z} \\ \partial/\partial z & 0 \end{bmatrix}.$$

Let  $\mathcal{L}(L^2(S))$  be the set of all bounded operators on  $L^2$ -sections of  $S$ . Let  $C^*(M)$  be the Roe algebra of  $M$ , that is,  $C^*(M)$  is the completion in  $\mathcal{L}(L^2(S))$  of the  $*$ -algebra of all bounded integral operators on  $L^2(S)$  with a smooth kernel and finite propagation [11, p.191]. We collect some well-known properties of the Roe algebra which we use.

<sup>1</sup>Every orientable surfaces are spin [10, p.88].

**Proposition 1.3.** [9, 11] *We assume that  $M$ ,  $S$ , and  $D$  are as above. The following holds.*

- (1) *Let  $f \in C_0(\mathbb{R})$  be a continuous function on  $\mathbb{R}$  with vanishing at infinity and let  $\lambda \in \mathbb{R}$ . Define  $D' := D + \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}$ , then  $f(D') \in C^*(M)$ .*
- (2) *Let  $D^*(M)$  be the unital  $C^*$ -algebra generated by all pseudolocal operators on  $L^2(S)$  with finite propagation. Then  $C^*(M)$  is a closed  $*$ -bisided ideal of  $D^*(M)$ .*
- (3)  *$fu \sim 0$  and  $uf \sim 0$  for all  $u \in C^*(M)$  and  $f \in C_0(M)$ .*
- (4) *Let  $\varpi$  be the characteristic function of  $M^+$ . Then  $[\varpi, u] \sim 0$  for all  $u \in C^*(M)$ .*

By using above properties, we define a  $K_1$ -element.

**Proposition 1.4.** *Let  $\phi \in C^1(M; GL_l(\mathbb{C}))$  be a continuously differentiable map from  $M$  to general linear group  $GL_l(\mathbb{C})$ . We assume that  $\|\phi\| < \infty$ ,  $\|\text{grad}(\phi)\| < \infty$  and  $\|\phi^{-1}\| < \infty$ . Define*

$$u_\phi := (D + \epsilon)^{-1} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} (D + \epsilon),$$

then  $u_\phi - \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} \in M_l(C^*(M))$ .

*Proof.* It suffices to show the  $l = 1$  case. Firstly,  $(D + \epsilon)^{-1} \in C^*(M)$  and  $\|(D + \epsilon)^{-1}\| \leq 2$  since  $(D + \epsilon)^{-1} = (D^2 + 1)^{-1}(D + \epsilon) \in C^*(M)$ . On the other hand,

$$\begin{aligned} u_\phi \sigma &= (D + \epsilon)^{-1} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & D^- \\ D^+ & -1 \end{bmatrix} \sigma \\ &= (D + \epsilon)^{-1} \begin{bmatrix} \phi & D^- \phi - D^- \phi + \phi D^- \\ D^+ & -1 \end{bmatrix} \sigma \\ &= (D + \epsilon)^{-1} \left( (D + \epsilon) \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + \begin{bmatrix} \phi - 1 & [\phi, D^-] \\ 0 & \phi - 1 \end{bmatrix} \right) \sigma \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} \sigma + (D + \epsilon)^{-1} \begin{bmatrix} \phi - 1 & -c(\text{grad}(\phi))^- \\ 0 & \phi - 1 \end{bmatrix} \sigma, \end{aligned}$$

for any  $\sigma \in C_c^\infty(S)$ , where  $c(\text{grad}(\phi))^-$  is the negative part of the Clifford action of  $\text{grad}(\phi)$ . So

$$\|u_\phi \sigma\|_{L^2} \leq 3(\|\phi\| + \|\text{grad}(\phi)\| + 1)\|\sigma\|_{L^2}.$$

Thus  $u_\phi$  can be extended uniquely as a bounded operator on  $L^2(S)$  since  $C_c^\infty(S)$  is dense in  $L^2(S)$ , and  $u_\phi - \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} \in C^*(M)$ .  $\square$

Let  $C_b(M)$  be the set of all bounded continuous functions on  $M$ . We assume that  $C_b^*(M) := C^*(M) + C_b(M)$  is a unital  $C^*$ -subalgebra of  $\mathcal{L}(L^2(S))$ . By this proposition,  $u_\phi$  is invertible in  $M_l(C_b^*(M))$  with  $(u_\phi)^{-1} = u_{\phi^{-1}}$ . So we can consider  $[u_\phi] \in K_1(C_b^*(M))$ .

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<sup>2</sup> $T \in \mathcal{L}(L^2(S))$  is pseudolocal if  $[f, T] \sim 0$  for all  $f \in C_0(M)$ , that is,  $[f, T]$  is compact.

**1.2. Cyclic cocycle and pairing.** Let  $\varpi$  be the characteristic function of  $M^+$ , and  $\chi := 2\varpi - 1$ . We note that  $[\chi, u]$  is a compact operator for all  $u \in C_b^*(M)$  since  $[\chi, f] = 0$  for all  $f \in C_b(M)$ . We define the Banach algebra

$$\mathcal{A}_b := \{A \in C_b^*(M) ; [\chi, A] \text{ is a trace class operator}\}$$

with norm  $\|A\|_{\mathcal{A}_b} := \|A\| + \|[\chi, A]\|_1$ , where  $\|\cdot\|$  is a operator norm on  $L^2(S)$  and  $\|\cdot\|_1$  is a trace norm. We define a cyclic cocycle on  $\mathcal{A}_b$  and take the pairing of it with  $K_1(C_b^*(M))$ .

**Proposition 1.5.** [11, Proposition 1.6] *For any  $A, B \in \mathcal{A}_b$ , we define*

$$\zeta(A, B) := \frac{1}{4} \text{Tr}(\chi[\chi, A][\chi, B]).$$

*Then  $\zeta$  is a cyclic one-cocycle on  $\mathcal{A}_b$ . We call that  $\zeta$  is the Roe cocycle.*

To take Connes' pairing of the Roe cocycle  $\zeta$  with  $K_1(C_b^*(M))$ , we use a next fact.

**Proposition 1.6.** [4, p.92]  *$\mathcal{A}_b$  is dense and closed under holomorphic functional calculus in  $C_b^*(M)$ . So the inclusion  $i : \mathcal{A}_b \rightarrow C_b^*(M)$  induces the isomorphism  $i_* : K_1(\mathcal{A}_b) \cong K_1(C_b^*(M))$ .*

*Proof.* Let  $\mathcal{X}$  be the  $*$ -algebra of all integral operators on  $L^2(S)$  with a smooth kernel and finite propagation speed. Then,  $\mathcal{X}_b := \mathcal{X} + C_b(M)$  is a dense subalgebra in  $C_b^*(M)$  and  $\mathcal{X}_b \subset \mathcal{A}_b$  [11, Proposition 1.6]. So  $\mathcal{A}_b$  is dense in  $C_b^*(M)$ .

The rest of proof is in [4, p.92].  $\square$

Using this proposition, we can take the pairing of the Roe cocycle with  $K_1(C_b^*(M))$  through the isomorphism  $i_* : K_1(\mathcal{A}_b) \cong K_1(C_b^*(M))$  as follows:

**Definition 1.7.** [4, p.109] *Define the map*

$$\langle \cdot, \zeta \rangle : K_1(C_b^*(M)) \rightarrow \mathbb{C}$$

*by  $\langle [u], \zeta \rangle := \frac{1}{8\pi i} \sum_{i,j} \zeta((u^{-1})_{ji}, u_{ij})$ , where we assume  $[u]$  is represented by a element of  $GL_n(\mathcal{A}_b)$  and  $u_{ij}$  is the  $(i, j)$ -component of  $u$ . We note that this is Connes' pairing of cyclic cohomology with  $K$ -theory, and  $\frac{1}{8\pi i}$  is a constant of the pairing.*

The goal of this paper is to prove that the result of this pairing with  $[u_\phi]$  is the Fredholm index of a Toeplitz operator.

**1.3. Toeplitz operators.** We review Toeplitz operators on  $S^1$  to fix notations.

**Proposition 1.8.** [6] *Let  $\phi \in C(S^1; GL_l(\mathbb{C}))$  be a continuous map from  $S^1$  to  $GL_l(\mathbb{C})$ . Define  $\mathcal{H} := \text{Span}_{\mathbb{C}}\{e^{ikx}; k = 0, 1, 2, \dots\} \subset L^2(S^1)$ <sup>3</sup> and let  $P : L^2(S^1) \rightarrow \mathcal{H}$  be the projection. Then for any  $f \in \mathcal{H}^l$ , we define Toeplitz operator  $T_\phi : \mathcal{H}^l \rightarrow \mathcal{H}^l$  by  $T_\phi f := P\phi f$ . Then  $T_\phi$  is a Fredholm operator and  $\text{index}(T_\phi) = -\deg(\det(\phi))$ , where  $\deg(\det(\phi))$  is the degree of the map  $\det(\phi) : S^1 \rightarrow \mathbb{C}^\times$ .*

We note that the Hardy space  $\mathcal{H}$  is a positive eigenspace of  $-i\partial/\partial x$ , which is a Dirac operator on  $S^1$ . See also [2, p.160].

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<sup>3</sup> $\mathcal{H}$  is called the Hardy space.

1.4. **Main theorem.** Using above notation, we state our main theorem as follows.

**Theorem 1.9.** *We also denote by  $\phi$  the restriction on  $N$  of  $\phi$ . Then the following formula holds:*

$$\langle [u_\phi], \zeta \rangle = -\frac{1}{8\pi i} \text{index}(T_\phi).$$

By the index theorem of Toeplitz operators (Proposition 1.8), right hand side of this formula is calculated by some geometric invariant of the mapping degree. So this theorem is a kind of index theorem (see also next section):

**Corollary 1.10.** *Using above notation,*

$$\text{index}(\varpi u_\phi \varpi : \varpi(L^2(S))^l \rightarrow \varpi(L^2(S))^l) = -\deg(\det(\phi)).$$

## 2. THE PAIRING AND THE FREDHOLM INDEX

To prove Theorem 1.9, we firstly describe  $\zeta(u^{-1}, u) = \sum_{i,j} \zeta((u^{-1})_{ji}, u_{ij})$  by the Fredholm index of some Fredholm operator.

**Proposition 2.1.** [5, IV.1] *For any  $u \in GL_n(\mathcal{A}_b)$ ,*

$$\zeta(u^{-1}, u) = -\text{index}(\varpi u \varpi : \varpi(L^2(S))^n \rightarrow \varpi(L^2(S))^n).$$

*Proof.* Since  $u \in GL_n(\mathcal{A}_b)$  and

$$\varpi - \varpi u^{-1} \varpi u \varpi = -\varpi[\varpi, u^{-1}][\varpi, u]\varpi,$$

so  $\varpi - \varpi u^{-1} \varpi u \varpi$  and  $\varpi - \varpi u \varpi u^{-1} \varpi$  are trace class operators on  $\varpi(L^2(S))^n$ . Therefore we get

$$\text{index}(\varpi u \varpi : \varpi(L^2(S))^n \rightarrow \varpi(L^2(S))^n) = \text{Tr}(\varpi - \varpi u^{-1} \varpi u \varpi) - \text{Tr}(\varpi - \varpi u \varpi u^{-1} \varpi)$$

by [4, p.88]. So we get

$$\begin{aligned} \text{index}(\varpi u \varpi : \varpi(L^2(S))^n \rightarrow \varpi(L^2(S))^n) &= \frac{1}{4} \text{Tr}(\chi[\chi, u][\chi, u^{-1}]) \\ &= \frac{1}{4} \sum_{i,j} \text{Tr}(\chi[\chi, u_{ij}][\chi, (u^{-1})_{ji}]) = -\zeta(u^{-1}, u). \end{aligned}$$

□

By this proposition and homotopy invariance of Fredholm indices, we get the following formula of our pairing and the Fredholm index:

$$(1) \quad \langle [u_\phi], \zeta \rangle = -\frac{1}{8\pi i} \text{index}(\varpi u_\phi \varpi : \varpi(L^2(S))^l \rightarrow \varpi(L^2(S))^l).$$

So we shall calculate this Fredholm index.

## 3. THE $\mathbb{R} \times S^1$ CASE

In this section we shall prove the  $M = \mathbb{R} \times S^1$  case, and in the next section we shall reduce the general case to  $M = \mathbb{R} \times S^1$  case.

In this section we assume  $M = \mathbb{R} \times S^1$ .  $\mathbb{R} \times S^1$  is partitioned by  $(\mathbb{R}_+ \times S^1, \mathbb{R}_- \times S^1, S^1)$ , where  $\mathbb{R}_\pm := \{t \in \mathbb{R}; t \geq 0 \text{ (resp. } t \leq 0)\}$ . Then the Dirac operator  $D$  of  $S = \mathbb{R} \times S^1 \times \mathbb{C}^2$  is given by the following formula:

$$D = \begin{bmatrix} 0 & -\partial/\partial t - i\partial/\partial x \\ \partial/\partial t - i\partial/\partial x & 0 \end{bmatrix},$$

where  $(t, x) \in \mathbb{R} \times S^1$ . Moreover, for any  $\phi \in C^1(S^1; GL_l(\mathbb{C}))$ , define  $\phi(t, x) := \phi(x)$  on  $\mathbb{R} \times S^1$ .

**3.1. Homotopy.** To calculate  $\text{index}(\varpi u_\phi \varpi : \varpi(L^2(S))^l \rightarrow \varpi(L^2(S))^l)$ , we perturb this operator by a homotopy.

**Proposition 3.1.** *For any  $s \in [0, 1]$ , define*

$$D_s := \begin{bmatrix} 0 & -\partial/\partial t + s/2 - i\partial/\partial x \\ \partial/\partial t + s/2 - i\partial/\partial x & 0 \end{bmatrix} = D + \begin{bmatrix} 0 & s/2 \\ s/2 & 0 \end{bmatrix}$$

and  $u_{\phi,s} := (D_s + (1-s)\epsilon)^{-1} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} (D_s + (1-s)\epsilon)$ .

Then  $[0, 1] \ni s \mapsto u_{\phi,s} \in \mathcal{L}(L^2(S)^l)$  is continuous and  $u_{\phi,s} - \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} \in M_l(C^*(M))$ .

*Proof.* It suffices to show the  $l = 1$  case. We note that  $\|D_s f\|_{L^2} \geq s\|f\|_{L^2}/2$  for any  $f \in \text{domain}(D_s) = \text{domain}(D)$  and  $s \in (0, 1]$ . Moreover  $D_s$  is self-adjoint. Therefore the spectrum of  $D_s$  and  $(-s/2, s/2)$  are disjoint, especially  $D_1^{-1} \in \mathcal{L}(L^2(S))$ .

Since  $(D_s + (1-s)\epsilon)^2 = D_s^2 + (1-s)^2$ , so  $(D_s + (1-s)\epsilon)^{-1} \in C^*(M)$ . Therefore  $u_{\phi,s}$  is well-defined as a densely defined closed operator of  $\text{domain}(u_{\phi,s}) = \text{domain}(D)$ . By similar proof of Proposition 1.4,

$$u_{\phi,s} = \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + (D_s + (1-s)\epsilon)^{-1} \begin{bmatrix} (1-s)(\phi - 1) & i\partial\phi/\partial x \\ 0 & (1-s)(\phi - 1) \end{bmatrix}$$

$$\text{and } u_{\phi,s} - \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} \in C^*(M).$$

Next we show  $\|u_{\phi,s} - u_{\phi,s'}\| \rightarrow 0$  as  $s \rightarrow s'$  for all  $s' \in [0, 1]$ . First, we show that  $\{\|(D_s + (1-s)\epsilon)^{-1}\|\}_{s \in [0,1]}$  is a bounded set. Set  $f_s(x) := \left| \frac{x}{x^2 + (1-s)^2} \right|$  and  $g_s(x) := \left| \frac{1}{x^2 + (1-s)^2} \right|$ . By fundamental calculus, we can show that

$$\sup_{|x| \geq s/2} |f_s(x)| \leq \frac{5}{2} \quad \text{and} \quad \sup_{|x| \geq s/2} |g_s(x)| \leq \frac{5}{4}.$$

Therefore

$$\begin{aligned} \|(D_s + (1-s)\epsilon)^{-1}\| &\leq \|(D_s^2 + (1-s)^2)^{-1} D_s\| + \|(1-s)(D_s^2 + (1-s)^2)^{-1}\| \\ &\leq \sup_{|x| \geq s/2} |f_s(x)| + \sup_{|x| \geq s/2} |g_s(x)| \leq 15/4 \end{aligned}$$

for all  $s \in [0, 1]$ .

On the other hand,

$$\begin{aligned} u_{\phi,s} - u_{\phi,s'} &= \{(D_s + (1-s)\epsilon)^{-1} - (D_{s'} + (1-s')\epsilon)^{-1}\} \begin{bmatrix} (1-s)(\phi - 1) & i\phi' \\ 0 & (1-s)(\phi - 1) \end{bmatrix} \\ &\quad + (D_{s'} + (1-s')\epsilon)^{-1} \begin{bmatrix} (s' - s)(\phi - 1) & 0 \\ 0 & (s' - s)(\phi - 1) \end{bmatrix} \end{aligned}$$

and the second term converges to 0 in operator norm topology as  $s \rightarrow s'$ , so we should only show that  $\|(D_s + (1-s)\epsilon)^{-1} - (D_{s'} + (1-s')\epsilon)^{-1}\| \rightarrow 0$  as  $s \rightarrow s'$ . But

this is proved by above uniformly boundness as follows:

$$\begin{aligned}
& \| (D_s + (1-s)\epsilon)^{-1} - (D_{s'} + (1-s')\epsilon)^{-1} \| \\
&= \| (D_s + (1-s)\epsilon)^{-1} ((s-s')\epsilon + D_{s'} - D_s) (D_{s'} + (1-s')\epsilon)^{-1} \| \\
&\leq \frac{3}{2} |s-s'| \| (D_{s'} + (1-s')\epsilon)^{-1} \| \| (D_s + (1-s)\epsilon)^{-1} \| \\
&\leq 32 |s-s'| \rightarrow 0.
\end{aligned}$$

□

By this proposition, we get

$$\text{index}(\varpi u_\phi \varpi) = \text{index}(\varpi u_{\phi,0} \varpi) = \text{index}(\varpi u_{\phi,1} \varpi).$$

Since

$$\varpi u_{\phi,1} \varpi = \begin{bmatrix} \varpi & 0 \\ 0 & \varpi(-\partial/\partial t + 1/2 - i\partial/\partial x)^{-1} \phi(-\partial/\partial t + 1/2 - i\partial/\partial x) \varpi \end{bmatrix} \left( =: \begin{bmatrix} \varpi & 0 \\ 0 & \mathcal{T}_\phi \end{bmatrix} \right),$$

so  $\text{index}(\varpi u_\phi \varpi)$  equals to

$$\text{index} \left( \mathcal{T}_\phi : \varpi(L^2(\mathbb{R})) \otimes L^2(S^1)^l \rightarrow \varpi(L^2(\mathbb{R})) \otimes L^2(S^1)^l \right),$$

where we assume that  $\varpi$  is the characteristic function of  $\mathbb{R}_+$ .

**3.2. The Hilbert transformation.** Let  $\mathcal{F}$  be the Fourier transformation:

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.$$

Let  $H$  be the Hilbert transformation <sup>4</sup>:

$$Hf(t) := \frac{i}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{t-y} dy,$$

where p.v. is Cauchy's principal value. Then we denote by  $\hat{P} : L^2(\mathbb{R}) \rightarrow \mathcal{H}_-$  the projection to the  $-1$ -eigenspace  $\mathcal{H}_-$  of  $H$ , that is,  $\hat{P} := \frac{1}{2}(1 - H)$ . Since  $\mathcal{F}$  is a invertible operator from  $\varpi(L^2(\mathbb{R}))$  to  $\mathcal{H}_-$ , so

$$\begin{aligned}
& \text{index} \left( \mathcal{T}_\phi : \varpi(L^2(\mathbb{R})) \otimes L^2(S^1) \rightarrow \varpi(L^2(\mathbb{R})) \otimes L^2(S^1) \right) \\
&= \text{index} \left( \mathcal{F} \mathcal{T}_\phi \mathcal{F}^{-1} : \mathcal{H}_- \otimes L^2(S^1) \rightarrow \mathcal{H}_- \otimes L^2(S^1) \right).
\end{aligned}$$

Set

$$\hat{\mathcal{T}}_\phi := \mathcal{F} \mathcal{T}_\phi \mathcal{F}^{-1} = \hat{P}(-it + 1/2 - i\partial/\partial x)^{-1} \phi(-it + 1/2 - i\partial/\partial x) \hat{P}^*.$$

To calculate the Fredholm index of  $\hat{\mathcal{T}}_\phi$ , we use a basis of  $L^2(\mathbb{R})$  made by eigenvectors of Hilbert transformation.

**Proposition 3.2.** [14, Theorem 1] *Set  $\rho_n \in L^2(\mathbb{R})$  by*

$$\rho_n(t) := \frac{(t-i)^n}{(t+i)^{n+1}}.$$

<sup>4</sup>The coefficient  $i/\pi$  of the Hilbert transformation is usually  $1/\pi$ . But we use this coefficient  $i/\pi$  because of  $H^2 = 1$ .

Then  $\{\rho_n/\sqrt{\pi}\}$  is a orthonormal basis of  $L^2(\mathbb{R})$  and

$$H\rho_n = \begin{cases} \rho_n & \text{if } n < 0 \\ -\rho_n & \text{if } n \geq 0 \end{cases}.$$

By using this basis, we get  $\mathcal{H}_- = \text{Span}_{\mathbb{C}}\{\rho_n; n \geq 0\}$  and we can calculate following Fredholm indices, which is used in the next subsection.

**Lemma 3.3.** For any  $\alpha, \beta \neq 0$ ,  $\hat{P} \frac{t+i\beta}{t+i\alpha} \hat{P}^* \in \mathcal{L}(\mathcal{H}_-)$  is a Fredholm operator and

$$\text{index} \left( \hat{P} \frac{t+i\beta}{t+i\alpha} \hat{P}^* \right) = \begin{cases} 0 & \text{if } \alpha\beta > 0 \\ -1 & \text{if } \alpha > 0, \beta < 0 \\ 1 & \text{if } \alpha < 0, \beta > 0 \end{cases}.$$

*Proof.* Our proof of Fredholmness is similar to [3, p.99]. Let  $c : \mathbb{R} \rightarrow S^1(\subset \mathbb{C})$  be the Cayley transformation defined by  $c(t) := (t-i)(t+i)^{-1}$ . Define  $\Phi(g)(t) := (t+i)^{-1}g(c(t))$  for any  $g \in L^2(S^1)$ . Then  $\Phi : L^2(S^1) \rightarrow L^2(\mathbb{R})$  is a invertible bounded linear operator with  $\|\Phi\| = 1/\sqrt{2}$  and  $\Phi^{-1}(f)(z) = (c^{-1}(z) + i)f(c^{-1}(z))$  for all  $f \in L^2(\mathbb{R})$  and  $z \in S^1 \setminus \{1\}$ .

Since  $\Phi(e^{inx}) = \rho_n$ ,  $\hat{P}f\hat{P}^*$  is a Fredholm operator on  $\mathcal{H}_-$  for any  $f \in C^\infty(\mathbb{R}; \mathbb{C}^\times)$  with  $\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow -\infty} f(t) \in \mathbb{C}^\times$ . Now,

$$\left| \frac{t+i\beta}{t+i\alpha} \right|^2 = \frac{t^2 + \beta^2}{t^2 + \alpha^2} > 0$$

and  $\lim_{t \rightarrow \pm\infty} \frac{t+i\beta}{t+i\alpha} = 1$ . Therefore  $\hat{P} \frac{t+i\beta}{t+i\alpha} \hat{P}^*$  is a Fredholm operator.

We calculate  $\text{index}(\hat{P} \frac{t+i\beta}{t+i\alpha} \hat{P}^*)$ . Define  $\text{sgn}(\alpha) := \begin{cases} 1 & \text{if } \alpha > 0 \\ -1 & \text{if } \alpha < 0 \end{cases}$ . Then we define a homotopy of Fredholm operators from  $\hat{P} \frac{t+i\beta}{t+i\alpha} \hat{P}^*$  to  $\hat{P} \frac{t+i\text{sgn}(\beta)}{t+i\text{sgn}(\alpha)} \hat{P}^*$  by

$$\hat{P} \frac{t+i(s\beta + (1-s)\text{sgn}(\beta))}{t+i(s\alpha + (1-s)\text{sgn}(\alpha))} \hat{P}^*$$

for  $s \in [0, 1]$ . Therefore

$$\text{index} \left( \hat{P} \frac{t+i\beta}{t+i\alpha} \hat{P}^* \right) = \text{index} \left( \hat{P} \frac{t+i\text{sgn}(\beta)}{t+i\text{sgn}(\alpha)} \hat{P}^* \right) = \begin{cases} 0 & \text{if } \alpha\beta > 0 \\ -1 & \text{if } \alpha > 0, \beta < 0 \\ 1 & \text{if } \alpha < 0, \beta > 0 \end{cases}$$

by  $\mathcal{H}_- = \text{Span}_{\mathbb{C}}\{\rho_n; n \geq 0\}$ .  $\square$

**3.3. The special case.** Set  $\phi_k(x) = e^{ikx}$  on  $S^1$  for  $k \in \mathbb{Z}$ . In this subsection, we calculate

$$\text{index}(\hat{\mathcal{T}}_{\phi_k} : \mathcal{H}_- \otimes L^2(S^1) \rightarrow \mathcal{H}_- \otimes L^2(S^1))$$

of the special case.

**Proposition 3.4.**  $\text{index}(\varpi u_{\phi_k} \varpi) = \text{index}(\hat{\mathcal{T}}_{\phi_k}) = -k = \text{index}(T_{\phi_k})$ .

*Proof.* The first equality is proved in subsection 3.1 and 3.2, and the last equality is well known. So we should only show the second equality. Let  $E_\lambda := \mathbb{C}\{e^{i\lambda x}\}$  be the  $\lambda$ -eigenspace of  $-i\partial/\partial x$ . On  $\mathcal{H}_- \otimes E_\lambda$ , operator  $\hat{\mathcal{T}}_{\phi_k}$  acts as

$$\hat{P}(-it + 1/2 + \lambda + k)^{-1}(-it + 1/2 + \lambda)\hat{P}^* \otimes \phi_k$$



and  $\hat{\mathcal{T}}_{\phi_k}(\mathcal{H}_- \otimes E_\lambda)$  is contained in  $\mathcal{H}_- \otimes E_{\lambda+k}$ . Therefore

$$\begin{aligned} & \text{index}(\hat{\mathcal{T}}_{\phi_k}) \\ &= \sum_{\lambda=-\infty}^{\infty} \text{index} \left( \hat{P} \frac{t + i(\lambda + 1/2)}{t + i(\lambda + k + 1/2)} \hat{P}^* \otimes \phi_k : \mathcal{H}_- \otimes E_\lambda \rightarrow \mathcal{H}_- \otimes E_{\lambda+k} \right) \\ &= -k \end{aligned}$$

by Lemma 3.3.  $\square$

**3.4. The general case.** Let  $\phi \in C^1(S^1; GL_l(\mathbb{C}))$  be a general continuously differentiable map. We reduce  $\phi$  case to  $\phi_k$  case.

Since the Gram-Schmidt orthogonalization  $GL_l(\mathbb{C}) \rightarrow U(l)$  is a homotopy equivalence map, so this map induces the isomorphism on fundamental groups  $\pi_1(GL_l(\mathbb{C})) \cong \pi_1(U(l))$ . By the homotopy long exact sequence, this inclusion  $i : C(S^1; S^1) \rightarrow C(S^1; U(l))$  of  $i(f) := \begin{bmatrix} f & 0 \\ 0 & 1_{l-1} \end{bmatrix}$  induces the isomorphism on fundamental groups  $i_* : \pi_1(S^1) \rightarrow \pi_1(U(l))$  [7, Example 4.55]. Moreover  $\pi_1(S^1) \cong \mathbb{Z}$  is represented by  $\phi_k$  for all  $k \in \mathbb{Z}$ . Therefore  $\phi$  is homotopic to  $\begin{bmatrix} \phi_k & 0 \\ 0 & 1_{l-1} \end{bmatrix}$  in  $C(S^1; GL_l(\mathbb{C}))$  for some  $k \in \mathbb{Z}$ . We denote  $\psi_s$  by this homotopy. Moreover, since  $C^1(S^1)$  is dense and closed under holomorphic functional calculus in  $C(S^1)$ , so we can take this homotopy  $\psi_s$  in  $C^1(S^1; GL_l(\mathbb{C}))$ .

**Proposition 3.5.**  $u_{\psi_s}$  is a continuous path in  $\mathcal{L}(L^2(S)^l)$ .

*Proof.* By proof of Proposition 1.4,

$$u_{\psi_s} = \begin{bmatrix} 1 & 0 \\ 0 & \psi_s \end{bmatrix} + (D + \epsilon)^{-1} \begin{bmatrix} \psi_s - 1 & i\psi'_s \\ 0 & \psi_s - 1 \end{bmatrix}.$$

Therefore

$$\|u_{\psi_s} - u_{\psi_{s'}}\| \leq 3\|\psi_s - \psi_{s'}\|_{C^1} \rightarrow 0$$

as  $s \rightarrow s'$  since  $\psi_s$  is a path in  $C^1(S^1; GL_l(\mathbb{C}))$ .  $\square$

*Proof of Theorem 1.9 of  $M = \mathbb{R} \times S^1$  case.* By Proposition 3.5,

$$\text{index}(\varpi u_\phi \varpi) = \text{index} \left( \varpi \begin{bmatrix} u_{\phi_k} & 0 \\ 0 & 1_{l-1} \end{bmatrix} \varpi \right) = \text{index}(\varpi u_{\phi_k} \varpi)$$

for some  $k \in \mathbb{Z}$ . By Proposition 3.4,  $\text{index}(\varpi u_{\phi_k} \varpi) = \text{index}(T_{\phi_k})$ . Moreover, since  $\phi$  is homotopic to  $\begin{bmatrix} \phi_k & 0 \\ 0 & 1_{l-1} \end{bmatrix}$ , so  $\text{index}(T_{\phi_k}) = \text{index}(T_\phi)$ . Therefore  $\text{index}(\varpi u_\phi \varpi) = \text{index}(T_\phi)$ . We complete a proof of Theorem 1.9 by using (1) in section 2.  $\square$

#### 4. THE GENERAL TWO-MANIFOLD CASE

In this section we reduce the general two dimensional manifold case to the  $\mathbb{R} \times S^1$  case. Our argument is similar to Higson's argument in [8]. Firstly, we shall show cobordism invariance of the pairing.

**Lemma 4.1.** [8, Lemma 1.4] *Let  $(M^+, M^-, N)$  and  $(M^{+'}, M^{-'}, N')$  be two partitions of  $M$ . Then we assume these two partitions are cobordant, that is, the symmetric differences  $M^\pm \triangle M^{\mp'}$  are compact. Let  $\varpi$  and  $\varpi'$  be the characteristic function of  $M^+$  and  $M^{+'}$ , respectively. We assume  $\phi \in C^1(M; GL_l(\mathbb{C}))$  satisfies  $\|\phi\| < \infty$ ,  $\|\text{grad}(\phi)\| < \infty$  and  $\|\phi^{-1}\| < \infty$ . Then  $\text{index}(\varpi u_\phi \varpi) = \text{index}(\varpi' u_\phi \varpi')$ .*

*Proof.* It suffices to show the  $l = 1$  case. Since  $[\phi, \varpi] = 0$  and  $[u_\phi, \varpi] \sim 0$ , so

$$\begin{aligned} & \text{index}(\varpi u_\phi \varpi : \varpi(L^2(S)) \rightarrow \varpi(L^2(S))) \\ &= \text{index} \left( (1 - \varpi) \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + \varpi u_\phi \varpi : L^2(S) \rightarrow L^2(S) \right) \\ &= \text{index} \left( (1 - \varpi) \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + \varpi u_\phi : L^2(S) \rightarrow L^2(S) \right) \\ &= \text{index} \left( \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + \varpi v_\phi : L^2(S) \rightarrow L^2(S) \right), \end{aligned}$$

where we assume  $v_\phi = u_\phi - \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} \in C^*(M)$ . Therefore we should only show  $\varpi v_\phi \sim \varpi' v_\phi$ . Now, since  $M^\pm \triangle M^{\mp'}$  are compact, there exists  $f \in C_0(M)$  such that  $\varpi - \varpi' = (\varpi - \varpi')f$ . So  $\varpi v_\phi - \varpi' v_\phi = (\varpi - \varpi')f v_\phi \sim 0$ .  $\square$

Secondly, we shall reduce the general manifold case to the  $\mathbb{R} \times N$  case used by Higson's Lemma.

**Lemma 4.2.** [8, Lemma 3.1] *Let  $M_1$  and  $M_2$  be two partitioned oriented two dimensional complete Riemannian manifolds and let  $S_j$  be a Hermitian vector bundle over  $M_j$ . Let  $\varpi_j$  be the characteristic function of  $M_j^+$ . We assume that there exists an isometry  $\gamma : M_2^+ \rightarrow M_1^+$  which lifts an isomorphism  $\gamma^* : S_1|_{M_1^+} \rightarrow S_2|_{M_2^+}$ . We denote again by  $\gamma^* : \varpi_1(L^2(S_1)) \rightarrow \varpi_2(L^2(S_2))$  the Hilbert space isometry defined by  $\gamma$ . Then we assume  $u_j \in GL_l(C^*(M_j))$  and  $f_j \in GL_l(C_b(M_j))$  satisfy  $v_j := u_j - f_j \in M_l(C^*(M))$  and  $\gamma^* u_1 \varpi_1 \sim \varpi_2 u_2 \gamma^*$ . Then  $\text{index}(\varpi_1 u_1 \varpi_1) = \text{index}(\varpi_2 u_2 \varpi_2)$ .*

*Similarly, if there exists an isometry  $\gamma : M_2^- \rightarrow M_1^-$  which lifts an isomorphism  $\gamma^* : S_1|_{M_1^-} \rightarrow S_2|_{M_2^-}$  and  $\gamma^* u_1 \varpi_1 \sim \varpi_2 u_2 \gamma^*$ , then  $\text{index}(\varpi_1 u_1 \varpi_1) = \text{index}(\varpi_2 u_2 \varpi_2)$ .*

*Proof.* It suffices to show  $l = 1$  case. Let  $v : (1 - \varpi_1)(L^2(S_1)) \rightarrow (1 - \varpi_2)(L^2(S_2))$  be any invertible operator. Then  $V := \gamma^* \varpi_1 + v(1 - \varpi_1) : L^2(S_1) \rightarrow L^2(S_2)$  is also invertible operator. Hence

$$\begin{aligned} & V((1 - \varpi_1) + \varpi_1 u_1 \varpi_1) - ((1 - \varpi_2) + \varpi_2 u_2 \varpi_2)V \\ &= -\gamma^* \varpi_1 + \varpi_2 \gamma^* + \gamma^* \varpi_1 u_1 \varpi_1 - \varpi_2 u_2 \varpi_2 \gamma^* \\ &\sim \gamma^* u_1 \varpi_1 - \varpi_2 u_2 \gamma^* \sim 0. \end{aligned}$$

Therefore  $\text{index}(\varpi_1 u_1 \varpi_1) = \text{index}(\varpi_2 u_2 \varpi_2)$  since  $V$  is an invertible operator and  $\text{index}(\varpi_j u_j \varpi_j) = \text{index}((1 - \varpi_j) + \varpi_j u_j \varpi_j)$  for  $j = 1, 2$ .  $\square$

In our case, we use this Lemma as follows:

**Corollary 4.3.** *Let  $M_j$  be a Riemannian surface and let  $S_j$  be a graded spin bundle over  $M_j$  with grading  $\epsilon_j$ . We assume that there exists an isometry  $\gamma : M_2^+ \rightarrow M_1^+$  which defines the Hilbert space isometry  $\gamma^* : \varpi_1(L^2(S_1)) \rightarrow \varpi_2(L^2(S_2))$  as in Lemma 4.2 and satisfies  $D_2 \gamma^* \sim \gamma^* D_1$  and  $\epsilon_2 \gamma^* \sim \gamma^* \epsilon_1$  on  $\varpi_1(L^2(S_1))$ . Let  $\phi_j \in C^1(M_j; GL_l(\mathbb{C}))$  satisfies  $\|\phi_j\| < \infty$ ,  $\|\text{grad}(\phi_j)\| < \infty$  and  $\|\phi_j^{-1}\| < \infty$  as in Proposition 1.4 and  $\phi_1$  and  $\phi_2$  satisfy  $\phi_1(\gamma(x)) = \phi_2(x)$  for all  $x \in M_2^+$ . Then  $\text{index}(\varpi_1 u_{\phi_1} \varpi_1) = \text{index}(\varpi_2 u_{\phi_2} \varpi_2)$ .*

*Proof.* We should only show  $\gamma^* u_{\phi_1} \varpi_1 \sim \varpi_2 u_{\phi_2} \gamma^*$ . Let  $\varphi_1$  be a smooth function on  $M_1$  such that  $\text{Supp}(\varphi_1) \subset M_1^+$  and there exists a compact set  $K_1 \subset M_1$  such that  $\varphi_1 = \varpi_1$  on  $M_1 \setminus K_1$ . Define  $\varphi_2(x) := \varphi_1(\gamma(x))$  for all  $x \in M_2^+$  and  $\varphi_2 = 0$  on  $M_2^-$ . Then  $\varphi_2$  is a smooth function on  $M_2$  such that  $\text{Supp}(\varphi_2) \subset M_2^+$  and there exists a compact set  $K_2 \subset M_2$  such that  $\varphi_2 = \varpi_2$  on  $M_2 \setminus K_2$ . Set  $v_{\phi_j} = u_{\phi_j} - \begin{bmatrix} 1 & 0 \\ 0 & \phi_j \end{bmatrix}$ . Then  $\gamma^* v_{\phi_1} \varpi_1 \sim \gamma^* v_{\phi_1} \varphi_1$  and  $\varpi_2 v_{\phi_2} \gamma^* \sim \varphi_2 v_{\phi_2} \gamma^*$ . So if  $\gamma^* v_{\phi_1} \varphi_1 \sim \varphi_2 v_{\phi_2} \gamma^*$ , then

$$\gamma^* u_{\phi_1} \varpi_1 \sim \gamma^* v_{\phi_1} \varphi_1 + \gamma^* \begin{bmatrix} 1 & 0 \\ 0 & \phi_1 \end{bmatrix} \varpi_1 \sim \varphi_2 v_{\phi_2} \gamma^* + \varpi_2 \begin{bmatrix} 1 & 0 \\ 0 & \phi_2 \end{bmatrix} \gamma^* \sim \varpi_2 u_{\phi_2} \gamma^*.$$

So we shall show  $\gamma^* v_{\phi_1} \varphi_1 \sim \varphi_2 v_{\phi_2} \gamma^*$ . In fact,

$$\begin{aligned} & \gamma^* v_{\phi_1} \varphi_1 - \varphi_2 v_{\phi_2} \gamma^* \\ &= \gamma^* (D_1 + \epsilon_1)^{-1} \begin{bmatrix} \phi_1 - 1 & -c(\text{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \varphi_1 - \varphi_2 (D_2 + \epsilon_2)^{-1} \begin{bmatrix} \phi_2 - 1 & -c(\text{grad}(\phi_2))^- \\ 0 & \phi_2 - 1 \end{bmatrix} \gamma^* \\ &= \{\gamma^* (D_1 + \epsilon_1)^{-1} \varphi_1 - \varphi_2 (D_2 + \epsilon_2)^{-1} \gamma^*\} \begin{bmatrix} \phi_1 - 1 & -c(\text{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \\ &\sim \{\gamma^* \varphi_1 (D_1 + \epsilon_1)^{-1} - (D_2 + \epsilon_2)^{-1} \gamma^* \varphi_1\} \begin{bmatrix} \phi_1 - 1 & -c(\text{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \\ &= (D_2 + \epsilon_2)^{-1} \{(D_2 + \epsilon_2) \gamma^* \varphi_1 - \gamma^* \varphi_1 (D_1 + \epsilon_1)\} (D_1 + \epsilon_1)^{-1} \begin{bmatrix} \phi_1 - 1 & -c(\text{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \\ &\sim (D_2 + \epsilon_2)^{-1} \gamma^* [D_1, \varphi_1] (D_1 + \epsilon_1)^{-1} \begin{bmatrix} \phi_1 - 1 & -c(\text{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \\ &\sim 0 \end{aligned}$$

since  $\text{grad}(\varphi_1)$  has a compact support and  $[D_1, \varphi_1] = c(\text{grad}(\varphi_1))$ . Thus we get  $\gamma^* u_{\phi_1} \varpi_1 \sim \varpi_2 u_{\phi_2} \gamma^*$ . Therefore  $\text{index}(\varpi_1 u_{\phi_1} \varpi_1) = \text{index}(\varpi_2 u_{\phi_2} \varpi_2)$  by Lemma 4.2.  $\square$

*Proof of Theorem 1.9.* Firstly, let  $a \in C^\infty([-1, 1]; [-1, 1])$  satisfies

$$a(t) = \begin{cases} -1 & \text{if } -1 \leq t \leq -3/4 \\ 0 & \text{if } -2/4 \leq t \leq 2/4 \\ 1 & \text{if } 3/4 \leq t \leq 1 \end{cases}.$$

Let  $(-4\delta, 4\delta) \times N$  be a tubular neighborhood of  $N$  in  $M$  satisfies

$$\sup_{(t,x),(s,y) \in [-3\delta, 3\delta] \times N} |\phi(t, x) - \phi(s, y)| < \|\phi^{-1}\|^{-1}.$$

Define  $\psi(t, x) := \phi(4\delta a(t), x)$  on  $(-4\delta, 4\delta) \times N$  and  $\psi = \phi$  on  $M \setminus (-4\delta, 4\delta) \times N$ . Then  $\psi \in C^1(M; GL_l(\mathbb{C}))$  and  $\|\psi - \phi\| < \|\phi^{-1}\|^{-1}$ . Thus  $[0, 1] \ni t \mapsto \psi_t := t\psi + (1-t)\phi$  satisfies  $\psi_t \in C^1(M; GL_l(\mathbb{C}))$ ,  $\|\psi_t\| < \infty$ ,  $\|\text{grad}(\psi_t)\| < \infty$  and  $\|\psi_t^{-1}\| < \infty$ ,  $\|\psi_t - \psi_{t'}\| \rightarrow 0$  and  $\|\text{grad}(\psi_t) - \text{grad}(\psi_{t'})\| \rightarrow 0$  as  $t \rightarrow t' \in [0, 1]$ . Therefore we should only show the case of which  $\phi$  satisfies  $\phi(t, x) = \phi(0, x)$  on tubular neighborhood  $(-2\delta, 2\delta) \times N$ . By Lemma 4.1 we may change a partition of  $M$  to  $(M^+ \cup ([-\delta, 0] \times N), M^- \setminus ((-\delta, 0] \times N), \{-\delta\} \times N)$  without changing  $\text{index}(\varpi u_\phi \varpi)$ . Then by Corollary 4.3 we may change  $M^+ \cup ([-\delta, 0] \times N)$  to  $[-\delta, \infty) \times N$  without changing  $\text{index}(\varpi u_\phi \varpi)$  where  $\phi$  is changed to  $\phi(t, x) = \phi(x)$  for  $(t, x) \in (-\delta, 0] \times N$ . We denote by  $M' := ([-\delta, \infty) \times N) \cup (M^- \setminus ((-\delta, 0] \times N))$  this manifold. Then  $M'$  is partitioned by  $([-\delta, \infty) \times N, M^- \setminus ((-\delta, 0] \times N), \{-\delta\} \times N)$ . By using Lemma 4.1 and

Corollary 4.3 again, we may change  $M'$  to  $\mathbb{R} \times N$  without changing  $\text{index}(\varpi u_\phi \varpi)$  with similar argument as above where  $\phi(t, x) = \phi(x)$  for  $(t, x) \in \mathbb{R} \times N$ . Now we have changed  $M$  to  $\mathbb{R} \times N = \mathbb{R} \times S^1$ .  $\square$

## 5. EXAMPLE

In this section, we see an example of a partitioned manifold with which the pairing  $\langle [u_\phi], \zeta \rangle$  is not zero but it is not  $\mathbb{R} \times S^1$ .

Firstly, we define partitioned two-manifold. Let  $\Sigma_2$  be a closed Riemannian surface of genus two and let  $C$  and  $C'$  be two submanifolds of  $\Sigma_2$  which define generators of  $H_1(\Sigma_2; \mathbb{Z})$  as Figure 2. Moreover, we cut  $\Sigma_2$  along  $C$  and  $C'$  and embed to  $\mathbb{R}^3$  like Figure 3.

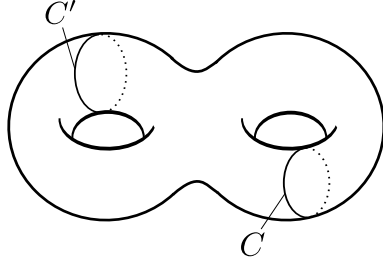


FIGURE 2.

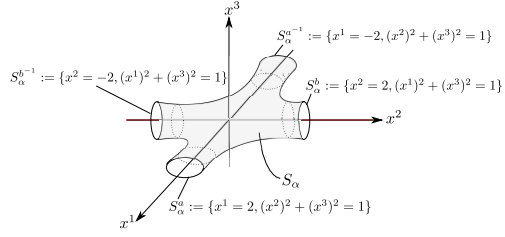


FIGURE 3.

Let  $F_2 := \langle a, b \rangle$  be the free group with two generators. For all  $\alpha \in F_2$ , we consider such surface  $S_\alpha$ . Then we assume  $S_\alpha$  is a oriented smooth manifold with Riemannian metric induced by  $\mathbb{R}^3$ . Moreover we assume  $T_\alpha^g(1/2)$  is a collar neighborhood of  $S_\alpha^g$  in  $S_\alpha$  for  $g \in \{a, a^{-1}, b, b^{-1}\}$ . Where for all  $\delta > 0$ , we define

$$\begin{aligned} T_\alpha^a(\delta) &:= \{2 - \delta < x^1 \leq 2, (x^2)^2 + (x^3)^2 = 1\}, \\ T_\alpha^{a^{-1}}(\delta) &:= \{-2 \leq x^1 < -2 + \delta, (x^2)^2 + (x^3)^2 = 1\}, \\ T_\alpha^b(\delta) &:= \{2 - \delta < x^2 \leq 2, (x^1)^2 + (x^3)^2 = 1\}, \text{ and} \\ T_\alpha^{b^{-1}}(\delta) &:= \{-2 \leq x^2 < -2 + \delta, (x^1)^2 + (x^3)^2 = 1\}. \end{aligned}$$

Let  $\phi_\alpha^g : \overline{T_\alpha^g(1/4)} \rightarrow [0, 1/4] \times S^1$  be the orientation preserving isometry defined by identification of  $\overline{T_\alpha^g(1/4)}$  with  $[0, 1/4] \times S^1$ . Define  $M := \bigsqcup_\alpha S_\alpha / \sim$ , where  $x \sim y$  if (i)  $x \in \overline{T_\alpha^g(1/4)}$  and  $y \in \overline{T_\beta^h(1/4)}$ , (ii)  $\alpha g = \beta h$  and (iii)  $\phi_\alpha^g(x) = \phi_\beta^h(y)$ . Then  $M$  is a oriented complete Riemannian manifold. Moreover,  $F_2$  acts on  $M$  freely. Let  $\pi : M \rightarrow M/F_2 = \Sigma_2$  be the quotient map of this action. We note that this manifold  $M$  forms like “the boundary of a fat picture” of Cayley graph of  $F_2$ .

Let  $N \subset \pi^{-1}(C)$  be a connected component of  $\pi^{-1}(C)$ , where  $C := \pi(S_\alpha^a)$ . Then  $M$  is separated two components by  $N$ . So we can define  $M^+$  and  $M^-$  which satisfy  $N = \partial M^-$ . Therefore  $M$  is a partitioned manifold.

On the other hand, there exists continuously differentiable map  $\varphi : \Sigma_2 \rightarrow GL_l(\mathbb{C})$  such that  $\deg(\det(\varphi|_C)) \neq 0$  since  $[C] \neq 0$ . For example, we choose  $\varphi : S^1 \rightarrow GL_l(\mathbb{C})$  such that  $\deg(\det(\varphi)) \neq 0$ , and we extend on  $T^2 = S^1 \times S^1$  trivially. Then we can define such  $\varphi$  on  $\Sigma_2$  through  $\Sigma_2 = T^2 \# T^2$ . Define  $\phi := \varphi \circ \pi$ , then  $\phi$  satisfies assumptions of Theorem 1.9.

In above setting,  $\deg(\det(\phi|_N)) = \deg(\det(\varphi|_C))$ . Therefore  $\langle [u_\phi], \zeta \rangle \neq 0$ .

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#### REFERENCES

- [1] Paul Baum and Ronald G. Douglas. *K* homology and index theory. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, Vol. 38 of *Proc. Sympos. Pure Math.*, pp. 117–173. Amer. Math. Soc., Providence, R.I., 1982.
- [2] Paul Baum and Ronald G. Douglas. Toeplitz operators and Poincaré duality. In *Toeplitz centennial (Tel Aviv, 1981)*, Vol. 4 of *Operator Theory: Adv. Appl.*, pp. 137–166. Birkhäuser, Basel, 1982.
- [3] B. Booss and D. D. Bleecker. *Topology and analysis*. Universitext. Springer-Verlag, New York, 1985. The Atiyah-Singer index formula and gauge-theoretic physics, Translated from the German by Bleecker and A. Mader.
- [4] Alain Connes. Noncommutative differential geometry. *Inst. Hautes Études Sci. Publ. Math.*, No. 62, pp. 257–360, 1985.
- [5] Alain Connes. *Noncommutative geometry*. Academic Press Inc., San Diego, CA, 1994.
- [6] I. C. Gohberg and M. G. Kreĭn. Systems of integral equations on the half-line with kernels depending on the difference of the arguments. *Uspehi Mat. Nauk (N.S.)*, Vol. 13, No. 2 (80), pp. 3–72, 1958.
- [7] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [8] Nigel Higson. A note on the cobordism invariance of the index. *Topology*, Vol. 30, No. 3, pp. 439–443, 1991.
- [9] Nigel Higson and John Roe. *Analytic K-homology*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. Oxford Science Publications.
- [10] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, Vol. 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.
- [11] John Roe. Partitioning noncompact manifolds and the dual Toeplitz problem. In *Operator algebras and applications, Vol. 1*, Vol. 135 of *London Math. Soc. Lecture Note Ser.*, pp. 187–228. Cambridge Univ. Press, Cambridge, 1988.
- [12] John Roe. *Elliptic operators, topology and asymptotic methods*, Vol. 395 of *Pitman Research Notes in Mathematics Series*. Longman, Harlow, second edition, 1998.
- [13] Tatsuki Seto. Toeplitz operators and the Roe-Higson type index theorem, 2014. arXiv:1405.4852.
- [14] J. A. C. Weideman. Computing the Hilbert transform on the real line. *Math. Comp.*, Vol. 64, No. 210, pp. 745–762, 1995.

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